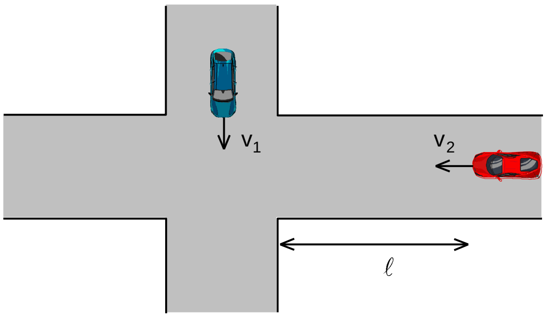
$$-\frac{\gamma Mm}{r}=\frac{mv^2}{2}-\frac{\gamma Mm}{R}\Rightarrow v=\pm\sqrt{2\left(\frac{\gamma M}{R}-\frac{\gamma M}{r}\right)}\Rightarrow v=-\sqrt{2\gamma M\left(\frac{r-R}{rR}\right)}$$

$$dt=\frac{dR}{v}=\frac{dR}{-\sqrt{2\gamma M\left(\frac{r-R}{rR}\right)}}=-\sqrt{\frac{r}{2\gamma M}}\sqrt{\frac{R}{r-R}}dr\Rightarrow t=-\int_r^R\sqrt{\frac{r}{2\gamma M}}\sqrt{\frac{R}{r-R}}dr$$

$$R=r\sin^2 z\Rightarrow R=r=\infty \text{ ise } z=\frac{\pi}{2}\Rightarrow R=0 \text{ ise } z=0\Rightarrow dR=2r\sin z\cos z.dz$$

$$t=-\int_{\frac{\pi}{2}}^{0}\sqrt{\frac{r}{2\gamma M}}\sqrt{\frac{r\sin^2 z}{r-r\sin^2 z}}.2r\sin z\cos z.dz=2\sqrt{\frac{r^3}{2\gamma M}}\int_0^{\frac{\pi}{2}}\sin^2 zdz=2\sqrt{\frac{r^3}{2\gamma M}}\int_0^{\frac{\pi}{2}}\frac{(1-\cos 2z)dz}{2}=$$

$$=\sqrt{\frac{r^3}{2\gamma M}}\left(z-\frac{\sin 2z}{2}\right)\Bigg|_0^{\frac{\pi}{2}}=\frac{\pi}{2}\sqrt{\frac{r^3}{2\gamma M}}$$



$$y=v_1t\Rightarrow x=\ell-v_2t\Rightarrow z=\sqrt{(v_1t)^2+(\ell-v_2t)^2}\Rightarrow\frac{df}{dt}=2v_1t-(\ell-v_2t)v_2=0$$

$$f(t)=v_1^2t^2+\ell^2-2\ell v_2t+v_2^2t^2=\left(v_1^2+v_2^2\right)t^2-2\ell v_2t+\ell^2\Rightarrow-\frac{b}{2a}=-\frac{-2\ell v_2}{2\left(v_1^2+v_2^2\right)}=\frac{\ell v_2}{v_1^2+v_2^2}$$

$$t_{min}=\frac{\ell v_2}{v_1^2+v_2^2}$$

$$z=\sqrt{\left(v_1^2+v_2^2\right).\left(\frac{\ell v_2}{v_1^2+v_2^2}\right)^2-2\ell v_2.\frac{\ell v_2}{v_1^2+v_2^2}+\ell^2}=\sqrt{\ell^2-\frac{\ell^2v_2^2}{v_1^2+v_2^2}}=\frac{\ell v_1}{\sqrt{v_1^2+v_2^2}}$$

$$y = h + v_0 t\sin\theta - \frac{gt^2}{2} \Rightarrow x = v_0 t\cos\theta \Rightarrow \sqrt{z} = \frac{1}{2\sqrt{z}}$$

$$\frac{gt^2}{2} - v_0 t\sin\theta - h = 0 \Rightarrow t^2 - \frac{2v_0 t\sin\theta}{g} - \frac{2h}{g} = 0 \Rightarrow t = \frac{\frac{2v_0\sin\theta}{g} + \sqrt{\frac{4{v_0}^2\sin^2\theta}{g^2} + \frac{8h}{g}}}{2} = \frac{v_0\sin\theta}{g} + \sqrt{\frac{{v_0}^2\sin^2\theta}{g^2} + \frac{2h}{g}}$$

$$x = \left(\frac{v_0\sin\theta}{g} + \sqrt{\frac{{v_0}^2\sin^2\theta}{g^2} + \frac{2h}{g}}\right)v_0\cos\theta = \frac{{v_0}^2\sin\theta\cos\theta}{g} + \sqrt{\frac{{v_0}^2\sin^2\theta}{g^2} + \frac{2h}{g}}.v_0\cos\theta$$

$$x = \frac{{v_0}^2\sin\theta\cos\theta}{g} + \frac{{v_0}^2}{g}\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}.\cos\theta$$

$$f(\theta) = \sin\theta\cos\theta + \sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}.\cos\theta \Rightarrow \frac{df}{d\theta} = \left(\cos^2\theta - \sin^2\theta\right) + \frac{1}{2}\frac{2\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}}.\cos\theta - \sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}.\sin\theta = 0$$

$$\left(\cos^2\theta - \sin^2\theta\right) + \frac{\sin\theta\cos^2\theta - \left(\sin^2\theta + \frac{2gh}{{v_0}^2}\right)\sin\theta}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}} = 0 \Rightarrow \cos^2\theta - \sin^2\theta + \frac{\sin\theta\cos^2\theta - \sin^3\theta - \frac{2gh\sin\theta}{{v_0}^2}}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}} = 0$$

$$\cos^2\theta - \sin^2\theta + \frac{\sin\theta\left(\cos^2\theta - \sin^2\theta\right) - \frac{2gh\sin\theta}{{v_0}^2}}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}} = 0 \Rightarrow \cos^2\theta - \sin^2\theta + \frac{\sin\theta\left(\cos^2\theta - \sin^2\theta\right)}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}} = \frac{\frac{2gh\sin\theta}{{v_0}^2}}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}}$$

$$\left(\cos^2\theta - \sin^2\theta\right)\left(1 + \frac{\sin\theta}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}}\right) = \frac{\frac{2gh\sin\theta}{{v_0}^2}}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}} \Rightarrow \left(\cos^2\theta - \sin^2\theta\right)\frac{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}} + \sin\theta}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}} = \frac{\frac{2gh\sin\theta}{{v_0}^2}}{\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}}}$$

$$\left(\cos^2\theta - \sin^2\theta\right)\left(\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}} + \sin\theta\right) = \frac{2gh\sin\theta}{{v_0}^2} \Rightarrow \sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}} + \sin\theta = \frac{2gh\sin\theta}{{v_0}^2\left(\cos^2\theta - \sin^2\theta\right)}$$

$$\sqrt{\sin^2\theta + \frac{2gh}{{v_0}^2}} = \frac{2gh\sin\theta}{{v_0}^2\left(\cos^2\theta - \sin^2\theta\right)} - \sin\theta \Rightarrow \sin^2\theta + \frac{2gh}{{v_0}^2} = \left[\frac{2gh\sin\theta}{{v_0}^2\left(\cos^2\theta - \sin^2\theta\right)} - \sin\theta\right]^2 =$$

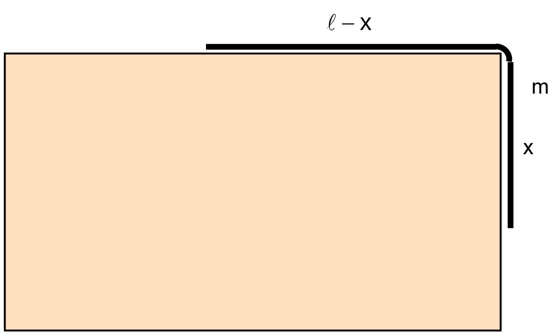
$$= \left[\frac{2gh\sin\theta}{{v_0}^2\left(\cos^2\theta - \sin^2\theta\right)}\right]^2 - \frac{4gh\sin^2\theta}{{v_0}^2\left(\cos^2\theta - \sin^2\theta\right)} + \sin^2\theta$$

$$\frac{2gh}{{v_0}^2} = \left[\frac{2gh\sin\theta}{{v_0}^2\cos^2\theta\left(1 - \frac{\sin^2\theta}{\cos^2\theta}\right)}\right]^2 - \frac{4gh\sin^2\theta}{{v_0}^2\cos^2\theta\left(1 - \frac{\sin^2\theta}{\cos^2\theta}\right)} = \frac{4g^2h^2\tan^2\theta}{{v_0}^4\cos^2\theta\left(1 - \tan^2\theta\right)^2} - \frac{4gh\tan^2\theta}{{v_0}^2\left(1 - \tan^2\theta\right)}$$

$$1 = \frac{2gh\tan^2\theta}{{v_0}^2\cos^2\theta\left(1 - \tan^2\theta\right)^2} - \frac{2\tan^2\theta}{1 - \tan^2\theta} = \frac{2gh\tan^2\theta}{{v_0}^2\frac{1}{1 + \tan^2\theta}\left(1 - \tan^2\theta\right)^2} - \frac{2\tan^2\theta}{1 - \tan^2\theta} = \frac{2gh\left(1 + \tan^2\theta\right)\tan^2\theta}{{v_0}^2\left(1 - \tan^2\theta\right)^2} - \frac{2\tan^2\theta}{1 - \tan^2\theta}$$

$$1 + \frac{2\tan^2\theta}{1 - \tan^2\theta} = \frac{2gh\left(1 + \tan^2\theta\right)\tan^2\theta}{{v_0}^2\left(1 - \tan^2\theta\right)^2} \Rightarrow \frac{1 + \tan^2\theta}{1 - \tan^2\theta} = \frac{2gh\left(1 + \tan^2\theta\right)\tan^2\theta}{{v_0}^2\left(1 - \tan^2\theta\right)^2} \Rightarrow 1 = \frac{2gh\tan^2\theta}{{v_0}^2\left(1 - \tan^2\theta\right)}$$

$${v_0}^2\left(1 - \tan^2\theta\right) = 2gh\tan^2\theta \Rightarrow {v_0}^2 - {v_0}^2\tan^2\theta = 2gh\tan^2\theta \Rightarrow {v_0}^2 = \left({v_0}^2 + 2gh\right)\tan^2\theta \Rightarrow \tan\theta = \frac{v_0}{\sqrt{{v_0}^2 + 2gh}}$$



$$\frac{mgx}{\ell} - F = \frac{mx}{\ell}a \Rightarrow F = \frac{m(\ell - x)}{\ell}a \Rightarrow \frac{gx}{\ell} = a$$

$$F = \frac{mg(\ell - x)x}{\ell^2} = \frac{mg(\ell x - x^2)}{\ell^2}$$

$$y = ax^2 + bx + c \Rightarrow x_T = -\frac{b}{2a}$$

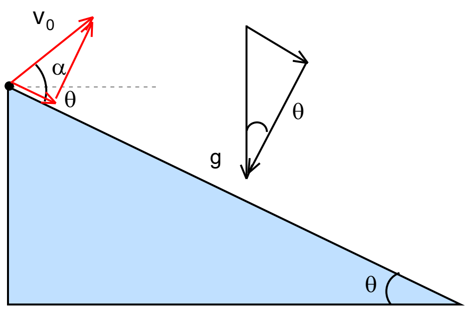
$$y = v_0 t\sin\theta - \frac{gt^2}{2} \Rightarrow x = v_0 t\cos\theta + \frac{at^2}{2}$$

$$0 = v_0 t\sin\theta - \frac{gt^2}{2} \Rightarrow t = \frac{2v_0\sin\theta}{g} \Rightarrow x = \frac{2v_0{}^2\sin\theta\cos\theta}{g} + \frac{a}{2}\frac{4v_0{}^2\sin^2\theta}{g^2} = \frac{2v_0{}^2}{g}\left(\sin\theta\cos\theta + \frac{a\sin^2\theta}{g}\right)$$

$$f(\theta) = \sin\theta\cos\theta + \frac{a\sin^2\theta}{g} \Rightarrow \frac{df}{d\theta} = \left(\cos^2\theta - \sin^2\theta\right) + \frac{2a\sin\theta\cos\theta}{g} = 0$$

$$\cos^2\theta + \frac{2a\sin\theta\cos\theta}{g} = \sin^2\theta \Rightarrow 1 + \frac{2a\sin\theta\cos\theta}{g\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} \Rightarrow 1 + \frac{2a\tan\theta}{g} = \tan^2\theta$$

$$\tan^2\theta - \frac{2a\tan\theta}{g} - 1 = 0 \Rightarrow \tan\theta = \frac{\frac{2a}{g} + \sqrt{\frac{4a^2}{g^2} + 4}}{2} = \frac{a}{g} + \sqrt{\frac{a^2}{g^2} + 1} = \frac{a + \sqrt{a^2 + g^2}}{g}$$



$$v_{0\parallel} = v_0\cos(\alpha+\theta) \Rightarrow v_{0\perp} = v_0\sin(\alpha+\theta) \Rightarrow t = \frac{2v_{0\perp}}{g_\perp} = \frac{2v_0\sin(\alpha+\theta)}{g\cos\theta}$$

$$x = v_{0\parallel}t + \frac{g_\parallel t^2}{2} = v_0\cos(\alpha+\theta).\frac{2v_0\sin(\alpha+\theta)}{g\cos\theta} + \frac{g\sin\theta}{2}\left[\frac{2v_0\sin(\alpha+\theta)}{g\cos\theta}\right]^2 =$$

$$= \frac{2v_0{}^2\sin(\alpha+\theta)\cos(\alpha+\theta)}{g\cos\theta} + \frac{2v_0{}^2\sin^2(\alpha+\theta)\sin\theta}{g\cos^2\theta} = \frac{2v_0{}^2}{g}\left[\frac{\sin(\alpha+\theta)\cos(\alpha+\theta)}{\cos\theta} + \frac{\sin^2(\alpha+\theta)\sin\theta}{\cos^2\theta}\right]$$

$$f(\theta) = \frac{\sin(\alpha+\theta)\cos(\alpha+\theta)}{\cos\theta} + \frac{\sin^2(\alpha+\theta)\sin\theta}{\cos^2\theta}$$

$$\frac{df}{d\theta} = \frac{\left[\cos^2(\alpha+\theta) - \sin^2(\alpha+\theta)\right]\cos\theta + \sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta}{\cos^2\theta} + \frac{2\sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta\cos^2\theta + 2\sin^2(\alpha+\theta)\sin^2\theta\cos\theta}{\cos^4\theta} = 0$$

$$\left[\cos^2(\alpha+\theta) - \sin^2(\alpha+\theta)\right]\cos\theta + \sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta + \frac{2\sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta\cos^2\theta + 2\sin^2(\alpha+\theta)\sin^2\theta\cos\theta}{\cos^2\theta} = 0$$

$$\cos^2(\alpha+\theta)\cos\theta - \sin^2(\alpha+\theta)\cos\theta + \sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta + \frac{2\sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta\cos\theta + 2\sin^2(\alpha+\theta)\sin^2\theta}{\cos\theta} = 0$$

$$\cos^2(\alpha+\theta)\cos^2\theta - \sin^2(\alpha+\theta)\cos^2\theta + \sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta\cos\theta + 2\sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta\cos\theta + 2\sin^2(\alpha+\theta)\sin^2\theta = 0$$

$$1 - \frac{\sin^2(\alpha+\theta)\cos^2\theta}{\cos^2(\alpha+\theta)\cos^2\theta} + \frac{\sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta\cos\theta}{\cos^2(\alpha+\theta)\cos^2\theta} + \frac{2\sin(\alpha+\theta)\cos(\alpha+\theta)\sin\theta\cos\theta}{\cos^2(\alpha+\theta)\cos^2\theta} + \frac{2\sin^2(\alpha+\theta)\sin^2\theta}{\cos^2(\alpha+\theta)\cos^2\theta} = 0$$

$$1 - \tan^2(\alpha+\theta) + 3\tan(\alpha+\theta)\tan\theta + 2\tan^2(\alpha+\theta)\tan^2\theta = 0$$